Some Comments on Likelihood Functions

The likelihood \mathcal{L} is a function of the parameters of a statistical model. It is used to estimate the values of and uncertainties on those parameters for a given set of measurements. For an ensemble of n measurements, the likelihood is defined as

$$\mathcal{L}(x;\theta) = \prod_{i=1}^{n} \mathcal{L}_{i}(x;\theta) = \prod_{i=1}^{n} f(x;\theta)$$

where $f(x; \theta)$ is the probability density function for the statistical model of interest. The best value of the parameters θ can be determined by maximizing the likelihood function, or equivalently, the log of the likelihood function.

$$\frac{\partial \ln \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial \theta} \ln \prod_{i=1}^{n} \mathcal{L}_{i}$$
$$= \frac{\partial}{\partial \theta} \sum_{i=1}^{n} \ln \mathcal{L}_{i}$$
$$= 0$$

Often this procedure is described instead as minimizing $-\ln \mathcal{L}$ (minimizing the minus log likelihood).

The log likelihood can be Taylor expanded about its minimum. Since $\partial \mathcal{L}/\partial \theta|_{\theta=\theta_{min}}=0$:

$$\ln \mathcal{L} = \ln \mathcal{L}_{min} + \frac{1}{2} \frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2} \Big|_{\theta = \theta_{min}} (\theta - \theta_{min})^2$$

$$2 (\ln \mathcal{L} - \ln \mathcal{L}_{min}) = \frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2} \Big|_{\theta = \theta_{min}} (\theta - \theta_{min})^2$$

In the limit of large n, the distribution \mathcal{L} (due to the central limit theorem) becomes Gaussian. Since for a Gaussian distribution a change in $2 \ln \mathcal{L}$ of one unit corresponds to a $1-\sigma$ variation in the parameter θ , the uncertainty on θ is given by:

$$\sigma_{\theta}^{2} \equiv \left\langle \left(\theta - \theta_{min}\right)^{2} \right\rangle = -\frac{1}{\frac{\partial^{2} \ln \mathcal{L}}{\partial \theta^{2}}}$$

Alternatively, the uncertainty on the estimated values of the parameters θ can be obtained by calculating the value of $\Delta\theta$ at which $-2 \ln \mathcal{L}$ increases by 1.0. In cases where $\ln \mathcal{L}$ is not parabolic, the uncertainties can be asymmetric.

The definitions above depend on the fact that the probability density function $f(x;\theta)$ is normalized over the region of x where measurements can occur

$$\int_{xmin}^{xmax} f(x;\theta)dx = 1$$

For this reason, likelihood fits are not sensitive to the value of n. It is possible to add a Poisson term to the likelihood function to include the number of events in the likelihood fit. When a fit includes such a term, it is called an *extended likelihood fit*.

An example

Suppose a set of measurements x_i are made in an experimental setup where the number of events as a function of x follows the distribution

$$N(x) = A + Bx$$
 for $0 < x < 10$

We would like to use the likelihood method to estimate the value of $\kappa \equiv A/B$. Let's see how to setup this problem.

The total number of events N_{Tot} can be determined

$$N_{Tot} = \int_0^{10} A + Bx$$

$$= \left(Ax + \frac{1}{2}Bx^2 \right) \Big|_0^{10}$$

$$= 10A + \frac{1}{2}(100B)$$

$$= 10A + 50B$$

and normalized probability density function $f(x;\theta)$ is

$$f(x; A, B) = \frac{1}{N_{Tot}} (A + Bx)$$

$$= \frac{1}{10A + 50B} (A + Bx)$$

$$= \frac{A}{10A + 50B} + \frac{Bx}{10A + 50B}$$

$$= 0.1 \left(\frac{\kappa}{\kappa + 5} + \frac{x}{\kappa + 5}\right)$$

The overall likelihood function is therefore

$$\mathcal{L}(x;\kappa) = \prod_{i=1}^{n} 0.1 \left(\frac{\kappa}{\kappa + 5} + \frac{x_i}{\kappa + 5} \right)$$

and the log likelihood is

$$\ln \left(\mathcal{L}(x;\kappa) \right) = \sum_{i=1}^{n} \ln \left(0.1 \left(\frac{\kappa}{\kappa + 5} + \frac{x_i}{\kappa + 5} \right) \right)$$
$$= \sum_{i=1}^{n} \ln \left(\left(\frac{\kappa}{\kappa + 5} + \frac{x_i}{\kappa + 5} \right) \right) + n \ln(0.1)$$

If the values x_i are known, then $-\ln(\mathcal{L}(x;\kappa))$ can be minimized with respect to κ . Because the last term in independent of x_i , it merely adds a constant term to the log likelihood and is irrelevant for the minimization.

There are many programs available to do such minimization (Root for example, has a good interface to the Minuit minimization package). However, you can also find the minimum by by seeing how $\ln(\mathcal{L}(x;\kappa))$ changes when κ is varied. An example root macro to generate fake data for the case A=1, B=2 and to use these data to determine κ can be found here:

http://physics.lbl.gov/shapiro/Physics226/myLikelihoodFit.C

The output of this macro is provided on the next page:

